# Notre Dame University
# Faculty of Engineering
# Previous Exams

## MAT213
## Calculus III
## Exam 1 - Fall 2010
## Duration: 60 minutes


**Contacts:**

**Joseph Hakmeh: 03 688 479**

**Joseph Matar: 70 158 443**

**1) (48 pts)** Evaluate the following integrals:

**a)** $\int e^t \sin t \, dt$

$u = e^t \qquad dv = \sin t \, dt$

$du = e^t dt \qquad v = -\cos t$ 2

$I = -e^t \cos t + \int e^t \cos t \, dt$ 2

$u = e^t \qquad dv = \cos t \, dt$

$du = e^t dt \qquad v = \sin t$

$I = -e^t \cos t + [e^t \sin t - \underbrace{\int e^t \sin t \, dt}_{I}]$ 2

$2I = e^t \sin t - e^t \cos t + C$

$I = \dfrac{e^t \sin t - e^t \cos t}{2} + C'$ 2

**b)** $\int \frac{dx}{\sqrt{8x - x^2 - 7}}$

$8x - x^2 - 7 = -(x^2 - 8x) - 7 = -(x^2 - 8x + 16 - 16) - 7$

$= -(x-4)^2 + 9$ 4

$J = \int \dfrac{dx}{\sqrt{9 - (x-4)^2}} = \sin^{-1}\left(\dfrac{x-4}{3}\right) + C$ 4

c) $\int \sin^3(x)\cos^2(x)\,dx = \int \sin^2 x \cos^2 x \sin x \, dx$ 2

$= \int (1-\cos^2 x)\cos^2 x \sin x \, dx$ 2

let $u = \cos x$

$du = -\sin x\, dx$ 1

$= \int (u^2 - u^4)(-du)$ 1

$= \int (u^4 - u^2)\, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$ 1 1

d) $\int \frac{(2x+4)}{x^3+4x}dx$

$$\frac{2x+4}{x^3+4x} = \frac{2x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$ 2

$$2x+4 = Ax^2 + 4A + Bx^2 + Cx$$

$$\begin{cases} A+B=0 \\ \boxed{C=2} \\ 4A=4 \end{cases}$$ 2

$\boxed{A=1}$ 1 $\rightarrow \boxed{B=-1}$ 1

$$\int \frac{2x+4}{x^3+4x}dx = \int \frac{1}{x}dx + \int \frac{-x+2}{x^2+4}dx$$

$$= \ln|x| - \int \frac{x}{x^2+4}dx + \int \frac{2}{x^2+4}dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+4) + \tan^{-1}\left(\frac{x}{2}\right) + C$$ 1 1 1

e) $\int \frac{dx}{x^2\sqrt{x^2+9}} = I$ (1)

8/1

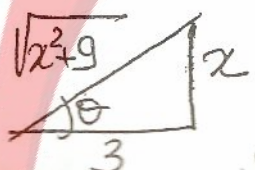
let $x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{3}\right) \quad \theta \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \rightarrow \sec\theta > 0$

$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3|\sec\theta| = 3\sec\theta$

$I = \int \frac{3\sec^2\theta\, d\theta}{9\tan^2\theta \cdot 3\sec\theta} = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$ (3)

$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$ (2)




$\csc\theta = \frac{hyp}{opp} = \frac{\sqrt{x^2+9}}{x}$

$= \frac{1}{9}\left[\frac{-1}{\sin\theta} + C\right]$ (1)

$= -\frac{1}{9}\csc\theta + C = -\frac{1}{9}\frac{\sqrt{x^2+9}}{x} + C$

f) $\int \frac{dx}{\sec x + \tan x} = \int \frac{1}{\sec x + \tan x} \cdot \boxed{\frac{\sec x - \tan x}{\sec x - \tan x}}\, dx$ (2)

$= \int \frac{\sec x - \tan x}{\sec^2 x - \tan^2 x}\, dx$ (1)

8/

$= \int \frac{\sec x - \tan x}{1}\, dx$ (1)

$= \int \sec x\, dx - \int \tan x\, dx$

$= \ln|\sec x + \tan x| - \ln|\sec x| + C$ (2) (2)

$= \ln|1 + \sin x| + C$

**2)(14 pts) a) (8 pts)** Evaluate: $\int \frac{6dx}{x^3+x^2-2x}$

$$\frac{6}{x^3+x^2-2x} = \frac{6}{x(x^2+x-2)} = \frac{6}{x(x+2)(x-1)} \quad 2$$

$$\frac{6}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1} \quad 1$$

$$A = \frac{6}{(0+2)(0-1)} = -3 \quad 1 \; ; \quad B = \frac{6}{(-2)(-3)} = 1 \quad 1 \; ;$$

$$C = \frac{6}{1\times(3)} = 2 \quad 1$$

$$\int \frac{6dx}{x^3+x^2-2x} = -3\int \frac{1}{x}dx + \int \frac{dx}{x+2} + 2\int \frac{dx}{x-1}$$

$$2 \quad = -3\ln|x| + \ln|x+2| + 2\ln|x-1| + C$$

**b) (6 pts)** Find: $\int_1^4 \frac{6dx}{x^3+x^2-2x}$

$$\int_1^4 \frac{6dx}{x^3+x^2-2x} = \lim_{a\to 1^+} \int_a^4 \frac{6dx}{x^3+x^2-2x} \quad 2$$

$$= \lim_{a\to 1^+} \Big[-3\ln|x| + \ln|x+2| + 2\ln|x-1|\Big]_a^4$$

$$2 \quad = \lim_{a\to 1^+} \left[-3\ln 4 + \ln 6 + 2\ln 3 + 3\ln a - \ln|a+2| - 2\ln|a-1|\right]$$

$$= -3\ln 4 + \ln 6 + 2\ln 3 + 0 - \ln 3 - 2\lim_{a\to 1^+} \ln|a-1|$$

2

**3) (18 pts)** Test the following integrals for convergence or divergence:

9

a) $\int_1^\infty \frac{\sin^2 x}{x^5}\,dx$

$0 \le \frac{\sin^2 x}{x^5} \le \frac{1}{x^5}$ 4

$\int_1^\infty \frac{1}{x^5}\,dx$ converges $(p=5>1)$ 3

By Direct Comp Test, $\int_1^\infty \frac{\sin^2 x}{x^5}\,dx$ converges 2

9

b) $\int_1^\infty \frac{x}{\sqrt{x^3+4x+2}}\,dx$

$$\lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^3+4x+2}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{x\sqrt{x}}{\sqrt{x^3+4x+2}} = \lim_{x\to\infty} \frac{x^{\frac{3}{2}}}{x^{\frac{3}{2}}} = 1$$ 4

$\frac{1}{\sqrt{x}}$ and $\frac{x}{\sqrt{x^3+4x+2}}$ both positive

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges $(p=\frac{1}{2}<1)$ 3

$\therefore \int_1^\infty \frac{x}{\sqrt{x^3+4x+2}}\,dx$ diverges by the Limit Comp Test 2

**4) (20 pts)** Consider the integral: $I_n = \int_0^\infty x^n e^{-x} dx$ , where $n$ is an integer , $n \geq 1$

a) **(8 pts)** Evaluate $I_1$

$$I_1 = \boxed{\lim_{b\to\infty} \int_0^b x e^{-x} dx} = \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b \quad 2 \quad 2 \quad 2$$

$$\int x e^{-x} dx$$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$-xe^{-x} + \int e^{-x} dx$

$-xe^{-x} - e^{-x}$

$$= \lim_{b\to\infty} \left[-be^{-b} - e^{-b} + 1\right]$$

$$= 0 - 0 + 1 = \boxed{1} \quad 2$$

b) **(6 pts)** Show that $I_n = n I_{n-1}$

$$I_n = \int_0^\infty x^n e^{-x} dx \qquad u = x^n \quad dv = e^{-x}dx$$

$$du = nx^{n-1}dx \quad v = -e^{-x}$$

$$I_n = \lim_{b\to\infty} \left[-x^n e^{-x}\right]_0^b + n\int_0^\infty x^{n-1} e^{-x} dx \quad 2$$

$$= \lim_{b\to\infty} \left[-b^n e^{-b} + 0\right] + n I_{n-1} \quad 2$$

$$= 1 \left(\lim_{b\to\infty} \frac{-b^n}{e^b}\right) + n I_{n-1} = n I_{n-1} \quad 1$$

$0$

c) **(6 pts)** Use part b) to evaluate $I_n$ and deduce it is a convergent integral for every $n \geq 1$

6

$$I_1 = 1 = 1!$$

$$I_n = n I_{n-1} = n(n-1) I_{n-2}$$

$$= n(n-1)(n-2) I_{n-3}$$

$$= n! \ldots$$

$\therefore I_n$ converges for every $n \geq 1$ since $I_n = n!$ (finite for every $n \geq 1$)